7.4

There's a Hole In My Function, Dear Liza

Graphical Discontinuities

LEARNING GOALS

In this lesson, you will:

- Sketch rational functions with removable discontinuities.
- Rewrite rational expressions.
- Compare removable discontinuities to vertical asymptotes.
- Identify domain restrictions of rational functions.

KEY TERM

- removable discontinuity

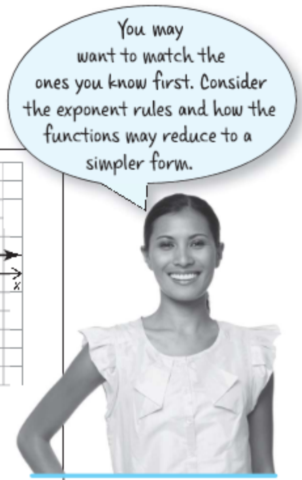
The ozone layer is a part of the atmosphere that contains high levels of ozone. Ozone absorbs the UV radiation from the sun that might otherwise be harmful to life on Earth. Recently, some experts believe that the ozone levels have been depleting over time. In particular, areas around the North and South poles have developed "ozone holes" which are allowing the harmful rays to enter our atmosphere.

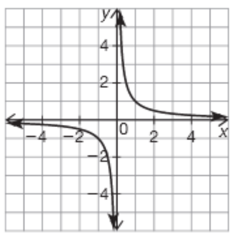
Why is the ozone depleting? What effects will this have on our environment?

PROBLEM 1 Mend Your Function, Dear Henry!

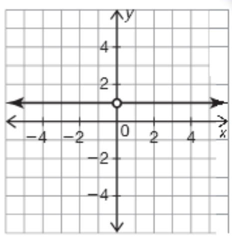
1. Without using a graphing calculator, match each rational function provided with the correct graph. Write the function below the graph.

$y = \frac{1}{x-2}$	$y = \frac{1}{x}$	$y = \frac{x^2}{x}$	$y = \frac{x-2}{x-2}$
$y = \frac{x^3}{x}$	$y = \frac{(x-2)^2}{x-2}$	$y = \frac{x}{x}$	$y = \frac{(x-2)^3}{x-2}$





Function:



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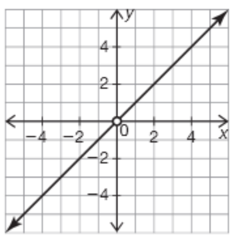

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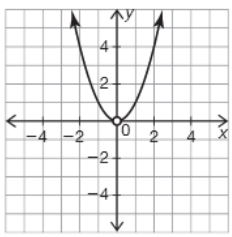

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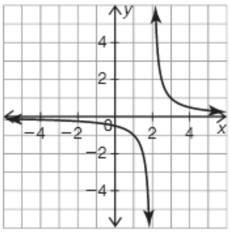
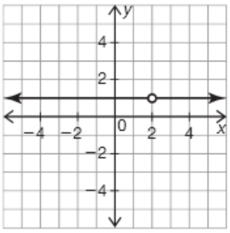
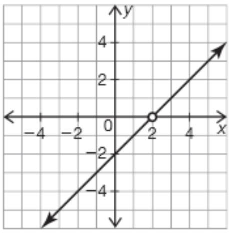
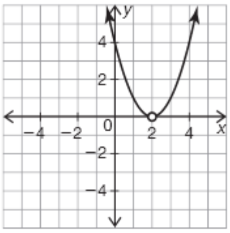
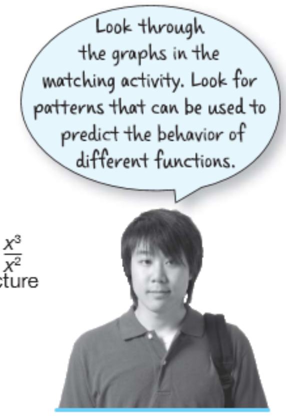
 Function:	 Function:
 Function:	 Function:

a. Which functions have asymptotes and which functions have "holes" in their graphs? Describe how the structure of the equation determines whether the function will have an asymptote or a "hole."

b. Compare the graphs of $y = \frac{1}{x-2}$ and $y = \frac{x-2}{x-2}$. How are they the same? How are they different? Describe how the structure of the equation determines these differences.

c. Compare the graphs of $y = \frac{x}{x}$ and $y = \frac{x-2}{x-2}$. How are they the same? How are they different? Describe the similarities and differences in the domain and range in terms of the structure of their equations.

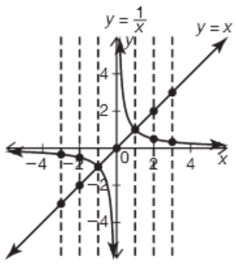


d. Without using a graphing calculator, describe the similarities and differences between the graphs of $y = \frac{x^3}{x^2}$ and $y = x$. Explain your reasoning in terms of the structure of the equations.

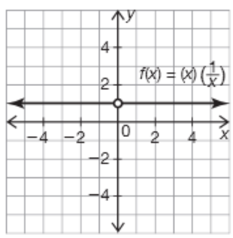
Many rational functions have "holes," or breaks, in the graphs instead of asymptotes. Let's analyze the structure of the function $y = \frac{x}{x}$ to determine why this function has a "hole" in the graph rather than a vertical asymptote at $x = 0$.

The function $y = \frac{x}{x}$ can be rewritten as the product of two factors: $y = (x)(\frac{1}{x})$. Looking at these reciprocal factors as separate functions reveals important characteristics.

x	-4	-3	-2	-1	0	1	2	3
$y = (x)\left(\frac{1}{x}\right)$	$(-4)\left(-\frac{1}{4}\right)$ 1	$(-3)\left(-\frac{1}{3}\right)$ 1	$(-2)\left(-\frac{1}{2}\right)$ 1	$(-1)\left(-\frac{1}{1}\right)$ 1	$(0)\left(\frac{1}{0}\right)$ und	$(1)\left(\frac{1}{1}\right)$ 1	$(2)\left(\frac{1}{2}\right)$ 1	$(3)\left(\frac{1}{3}\right)$ 1

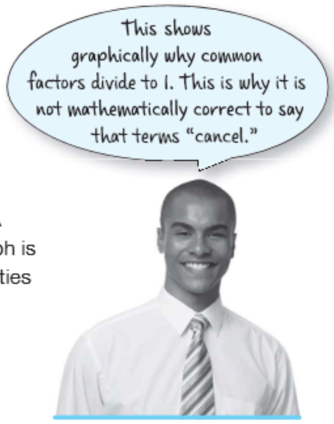


Graphical representation of each factor



Graphical representation of the product

Multiplying the outputs for each input reveals that $(x)\left(\frac{1}{x}\right) = 1$. This graph is a horizontal line that is undefined at $x = 0$. It is undefined at $x = 0$ because this is the value for which an asymptote exists for the part of the original equation, $\frac{1}{x}$. Similar reasoning can be used to show that for any function $f(x)$, $f(x) \cdot \frac{1}{f(x)} = 1$, with breaks in the graph for all undefined values where $f(x) = 0$. These breaks, or "holes," in the graph are called *removable discontinuities*. A **removable discontinuity** is a single point at which the graph is not defined. Vertical asymptotes and removable discontinuities must be listed as domain restrictions.



2. Henry and Liza each describe a different way to graph $y = \frac{x^3}{x^2}$.

Henry

I know any function multiplied by its reciprocal is 1.

I can rewrite the function as

$y = \frac{x^3}{x^2} = x \cdot \left(\frac{x^2}{x^2}\right)$.

This means that the output of $y = x$ is multiplied by $y = 1$ with a discontinuity at $x = 0$. The result is the line $y = x$ with a removable discontinuity at $(0, 0)$, so $x \neq 0$.

Liza

A removable discontinuity will exist anywhere that the denominator is 0 for the original function. In this case it is $(0, 0)$. I can use the exponent rules to simplify the function

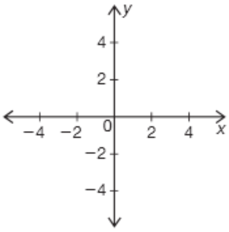
$$y = \frac{x^3}{x^2} = x^{3-2} = x^1.$$

Then I can just graph $y = x$ with a hole at $(0, 0)$, so $x \neq 0$.

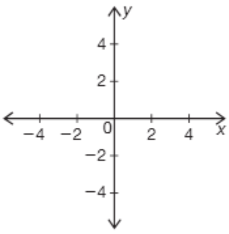
Which method do you prefer? Explain your reasoning.

3. Without using a graphing calculator, sketch the graph of each function. Be sure to note any asymptotes or holes in the graph.

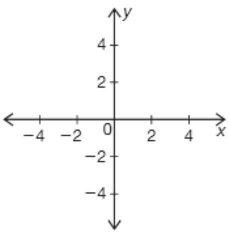
a. $y = \frac{2x^2}{x^2}$



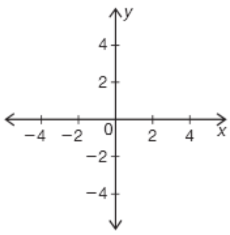
b. $y = \frac{x^2}{x^3}$



c. $y = \frac{x^4}{x^1}$



d. $y = \frac{-x^2}{x^4}$



PROBLEM 2 With What Shall I Mend The Function, Dear Liza?

1. Henry graphed the rational function $f(x) = \frac{x^2 + x - 6}{x - 2}$. Analyze his work.

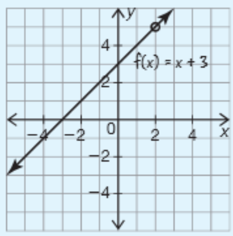
Henry

I know there is a domain restriction, so $x \neq 2$. I'm not sure if this is a vertical asymptote or a removable discontinuity, so I'm going to factor the numerator, if possible, to see if a common factor exists.

$$f(x) = \frac{x^2 + x - 6}{x - 2} = \frac{(x - 2)(x + 3)}{x - 2}$$
$$= \frac{1(x + 3)}{1} = x + 3$$

I know the output values of $\frac{(x - 2)}{(x - 2)} = 1$ with a discontinuity at $x = 2$. Therefore I can simply graph $f(x) = x + 3$. The removable discontinuity is at $(2, 2 + 3)$ and appears as a "hole" in the graph.

Note that the common factors do not "cancel." Many people use this term incorrectly to describe when factors divide to equal 1.



a. Why did Henry include an open circle at (2, 5) and *not* a vertical asymptote at $x = 2$?

b. Explain why $f(x) = \frac{x^2 + x - 6}{x - 2}$ can be rewritten as $f(x) = x + 3$.

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The graphs of rational functions will have either a removable discontinuity or a vertical asymptote for all domain values that result in division by 0. Simplifying rational expressions is similar to simplifying rational numbers: common factors divide to 1.

2. Analyze the table that shows similarities between rational numbers and rational functions.

		Rational Numbers	Rational Expressions
A common numerator and denominator divide to equal 1.	Examples	$\frac{5}{5} = 1$	$\frac{x}{x} = 1$
		$\frac{10.7}{10.7} = 1$	$\frac{5x}{5x} = 1$
		$\frac{0.025 + 0.016}{0.025 + 0.016} = 1$	$\frac{x + 5}{x + 5} = 1$
Common monomial factors divide to equal 1.	Examples	$\frac{5 \cdot 3}{5} = \frac{1 \cdot 3}{1}$ $= 3$	$\frac{5x}{5} = \frac{1 \cdot x}{1} = x$
		$\frac{4}{4 \cdot 6} = \frac{1}{1 \cdot 6} = \frac{1}{6}$	$\frac{x}{xz} = \frac{1}{1 \cdot z} = \frac{1}{z}$
Common binomial factors divide to equal 1.	Examples	$\frac{(5 + 3)(16 - 7)}{(5 + 3)} = \frac{1 \cdot (16 - 7)}{1}$ $= 16 - 7$	$\frac{(x + 5)(x - 4)}{(x + 5)} = \frac{1(x - 4)}{1}$ $= (x - 4)$
		$\frac{(9 - 4)}{(9 - 4)(9 + 5)} = \frac{1}{(9 + 5)}$	$\frac{x - 4}{(x - 4)(x + 5)} = \frac{1}{(x + 5)}$

a. Describe how simplifying rational numbers is similar to simplifying rational expressions.

b. Why is there a 1 in the numerator after simplifying $\frac{x}{xz} = \frac{1}{z}$?

c. For each example in the rational expressions column, list any restrictions on the domain.

3. Liza rewrites the rational expression as shown.

Liza

$$\frac{x^2 + 4x + 3}{4x + 3} = x^2$$

I divided out the common factors. The numerator and denominator each have a 4x and a 3, so I am left with the squared term.

Describe the error in Liza's reasoning.

4. Simplify each rational expression. List any restrictions on the domain.

a. $\frac{2x^2 - 8}{x - 2}$

b. $\frac{3xy - 3y}{x^2 - 1}$

c. $\frac{x^2 - 5x + 6}{3x - 9}$

d. $\frac{x^3 - 7x^2 - 18x}{3x^2 - 9x}$

e. $\frac{25x^2 - 9}{5x^2 - 12x - 9}$

f. $\frac{x^3 - 5x^2 - x + 5}{x^2 - 6x + 5}$

PROBLEM 3 Use Your Head, Dear Henry!

1. Determine whether the graph of the rational function has a vertical asymptote, a removable discontinuity, both, or neither. List the discontinuities and justify your reasoning.

   a. $j(x) = \frac{x + 2}{x(x + 2)}$

   b. $h(x) = \frac{x}{x + 5}$

   c. $j(x) = \frac{5}{5(x + 2)}$

   d. $j(x) = \frac{x + 2}{x^2 - 2x - 15}$

2. Write two examples of rational functions with one or more removable discontinuities. Explain your reasoning.

3. Write a unique function that has a vertical asymptote and a removable discontinuity. Explain your reasoning.

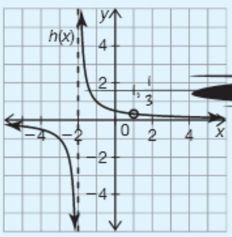
4. Liza graphed the rational function $h(x) = \frac{x-1}{x^2+x-2}$. Analyze her work.

Liza

I'm not sure where the asymptotes are, so I'm going to factor the denominator if possible.

$$h(x) = \frac{x-1}{x^2+x-2} = \frac{x-1}{(x-1)(x+2)}$$
$$= \frac{1}{x+2}.$$

I know there are domain restrictions at $x = 1$ and $x = -2$. The common factor $(x - 1)$ is in the numerator so $\frac{x-1}{x-1} = 1$. Therefore $x = 1$ is a removable discontinuity, while $x = -2$ is a vertical asymptote. I can quickly sketch $h(x) = \frac{1}{x+2}$ as a horizontal shift of $h(x) = \frac{1}{x}$ two units to the left. I know a discontinuity will exist at $\left(1, \frac{1}{1+2}\right)$, or $\left(1, \frac{1}{3}\right)$. A horizontal asymptote is at $y = 0$ and the y-intercept is $\left(0, \frac{1}{2}\right)$.



a. Summarize why $x = -2$ is a vertical asymptote while $x = 1$ appears as a "hole" in the graph.

b. Explain why the graph has a horizontal asymptote at $y = 0$.

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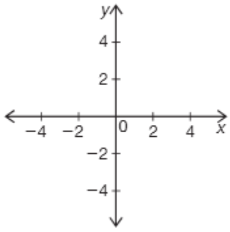
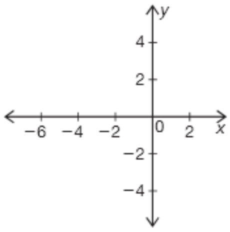
5. Sketch each function without the use of a graphing calculator. Identify any restrictions.

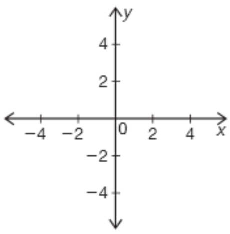
a. $f(x) = \dfrac{x + 2}{x^2 + 4x + 4}$



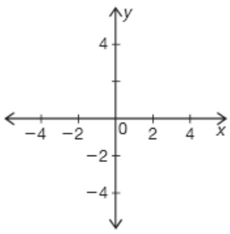
b. $g(x) = \dfrac{x}{x^2 + 3x}$



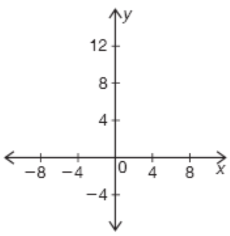
c. $h(x) = \dfrac{x}{x^3 - x}$



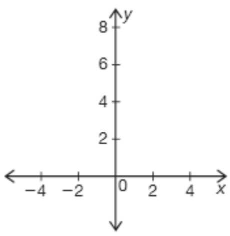
d. $m(x) = \dfrac{x^2 + 5x + 6}{x^2 + 5x + 6}$



e. $k(x) = \dfrac{x^2 - 3x - 15}{x - 5}$



f. $k(x) = \dfrac{x^3 + x^2 + 2x + 2}{x + 1}$



Talk the Talk

1. Describe the similarities and differences between rational numbers and rational expressions.

2. Describe the similarities and differences between vertical asymptotes and removable discontinuities.

3. Why is it incorrect to describe division by a common factor as "canceling out"?

Be prepared to share your solutions and methods.